New aspects of the Foldy-Wouthuysen and Cini-Touscheck transformations

1978 J. Phys. A: Math. Gen. 11 1185

(http://iopscience.iop.org/0305-4470/11/6/018)

J. Phys. A: Math. Gen., Vol. 11, No. 6, 1978. Printed in Great Britain

New aspects of the Foldy–Wouthuysen and Cini–Touscheck transformations

D Saroja†, K N Srinivasa Rao and A V Gopala Rao

Department of Physics, University of Mysore, Manasagangotri, Mysore 570 006, India

Received 6 September 1977, in final form 6 January 1978

Abstract. A procedure for obtaining the Cini–Touscheck (CT) analogue of any Foldy–Wouthuysen (FW)-type transformation is given. In the case of the free spin-$\frac{1}{2}$ particle, the proper roles of the FW, CT and rotational transformations in the construction of the simultaneous eigenstates of energy and helicity are discussed.

1. Introduction

In spite of the considerable interest shown (Foldy and Wouthuysen 1950, Krajcik and Nieto 1976, Foldy 1962, Rose 1961, Bjorken and Drell 1964, Schweber 1964, Messiah 1962, Roman 1965, Eisele 1969, Davydov 1965, Umezawa 1956, Cini and Touscheck 1958 and Bose *et al* 1959; see Krajcik and Nieto 1976 for other related references) in the free-particle FW and CT transformations, some interesting aspects of these transformations seem to have remained unnoticed in the literature. In this paper we show that the CT transformation (Cini and Touscheck 1958, Bose *et al* 1959) of the free-particle Dirac Hamiltonian may be written as a product of two FW transformations and that the same procedure is also applicable to *any* free-particle Dirac-type Hamiltonian. In particular we obtain, as new results, the CT analogue of the Melosh transformation (Melosh 1973, Weyers 1975) connecting constituent and current quarks, and the CT analogue of the Garrido–Pascual (GP) transformation (Garrido and Pascual 1959) of the Case Hamiltonian (Case 1954) for arbitrary spin. Secondly, we show how the FW and CT transformations can be used in a natural way for the construction of the simultaneous eigenstates of energy and helicity in respectively the Dirac–Pauli (DP) and the Weyl–Kramers (WK) representations of the Dirac matrices. The simultaneous eigenstates, in *any arbitrary representation*, on the other hand, result directly by coupling either one of these transformations with an appropriate rotation.

2. The relation between the CT and FW transformations

The FW transformation T which sends the free-particle Dirac Hamiltonian

$$H = \alpha_k p_k + \beta m \tag{1}$$

† Formerly CSIR Senior Research Fellow.

to the form

$$T^{+}HT=(p^{2}+m^{2})^{1/2}\beta; \qquad T^{+}=T^{-1} \tag{2}$$

is given by (Foldy and Wouthuysen 1950)

$$T=\exp\left(-\frac{\mathrm{i}}{2p}\gamma_{k}p_{k}\tan^{-1}\frac{p}{m}\right). \tag{3}$$

On passing to the limit as $m\to 0$, we obtain the zero-mass transformation

$$T_{0}\equiv\lim_{m\to 0}T=\exp\left(-\frac{\mathrm{i}\pi}{4p}\gamma_{k}p_{k}\right) \tag{4}$$

and we see from (1) and (2) that the zero-mass Hamiltonian $H_0=\alpha_k p_k$ is transformed into:

$$T_{0}^{+}\alpha_{k}p_{k}T_{0}=p\beta. \tag{5}$$

On dividing by p and multiplying by $(p^2+m^2)^{1/2}$, (5) becomes

$$T_{0}^{+}\left(\frac{\alpha_{k}p_{k}}{p}(p^{2}+m^{2})^{1/2}\right)T_{0}=(p^{2}+m^{2})^{1/2}\beta=T^{+}HT$$

so that

$$S^{+}HS=\frac{\alpha_{k}p_{k}}{p}(p^{2}+m^{2})^{1/2}$$

where

$$S=TT_{0}^{+}=\exp\left\{-\frac{\mathrm{i}}{2p}\gamma_{k}p_{k}\left[\tan^{-1}\left(\frac{p}{m}\right)-\frac{\pi}{2}\right]\right\}=\exp\left(\frac{\mathrm{i}}{2p}\gamma_{k}p_{k}\tan^{-1}\frac{m}{p}\right) \tag{6}$$

is precisely the CT transformation (Cini and Touscheck 1958, Bose *et al* 1959) of the free-particle Dirac Hamiltonian. This desired relation between S, T and T_0 can easily be applied to any FW-type transformation of a free Dirac-type Hamiltonian. As examples of this procedure, we now obtain the CT analogues of two FW-type transformations of considerable interest.

3. The CT analogue of the Melosh transformation

Recently Melosh (Melosh 1973, Weyers 1975) has used the partial FW transformation

$$T_{\mathrm{M}}=\exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^{3}x\, q^{+}(x)\left(\frac{\boldsymbol{\gamma}_{\perp}\cdot\boldsymbol{p}_{\perp}}{p_{\perp}}\tan^{-1}\frac{p_{\perp}}{m}\right)q(x)\right] \tag{7}$$

where $q(x)$ is the quantised local relativistic quark field operator,

$$\boldsymbol{\gamma}_{\perp}\cdot\boldsymbol{p}_{\perp}\equiv\gamma_{1}p_{1}+\gamma_{2}p_{2}, \qquad p_{\perp}\equiv(p_{1}^{2}+p_{2}^{2})^{1/2}$$

and γ_μ are the usual Dirac matrices, to connect the generators of the SU(6) algebras of the constituent and current quarks in the free-quark model (see Weyers 1975 for a recent review). This transformation T_{M} maps the free quark field Hamiltonian

$$H=\int \mathrm{d}^{3}x\, q^{+}(x)(\alpha_{k}p_{k}+\beta m)q(x) \tag{8}$$

into the form

$$T_{\mathrm{M}}^{+} H T_{\mathrm{M}} = \int \mathrm{d}^3x\, q^{+}(x)[\alpha_3 p_3 + (m^2 + p_{\perp}^2)^{1/2}\beta] q(x) \tag{9}$$

and is a partial FW transformation in the sense that replacing $\boldsymbol{\gamma}_{\perp} \cdot \boldsymbol{p}_{\perp}$ by $\gamma_k p_k$ and $p_{\perp}$ by p in T_{M} would make it identical to T of (3). In the limit $m \to 0$, as before, (9) reduces to

$$T_{\mathrm{M}_0}^{+} H_0 T_{\mathrm{M}_0} = \int \mathrm{d}^3x\, q^{+}(x)(\alpha_3 p_3 + p_{\perp}\beta) q(x) \tag{10}$$

where

$$H_0 = \int \mathrm{d}^3x\, q^{+}(x) \alpha_k p_k q(x) \tag{11}$$

and

$$T_{\mathrm{M}_0} = \exp\left(-\frac{\mathrm{i}\pi}{4} \int \mathrm{d}^3x\, q^{+}(x) \frac{\boldsymbol{\gamma}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}} q(x)\right). \tag{12}$$

Cancelling the term containing $\alpha_3 p_3$ on both sides of (10) ($\alpha_3 p_3$ commutes with T_{M_0}), dividing it by $p_{\perp}$ and multiplying by $(m^2 + p_{\perp}^2)^{1/2}$, we get

$$T_{\mathrm{M}_0}^{+}\left[\int \mathrm{d}^3x\, q^{+}(x)\left(\frac{\boldsymbol{\alpha}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}}(m^2 + p_{\perp}^2)^{1/2}\right) q(x)\right] T_{\mathrm{M}_0} = \int \mathrm{d}^3x\, q^{+}(x)[(m^2 + p_{\perp}^2)^{1/2}\beta] q(x).$$

If we now add the term $\int \mathrm{d}^3x\, q^{+}(x)\alpha_3 p_3 q(x)$ to both sides we get

$$T_{\mathrm{M}_0}^{+}\left[\int \mathrm{d}^3x\, q^{+}(x)\left(\frac{\boldsymbol{\alpha}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}}(m^2 + p_{\perp}^2)^{1/2} + \alpha_3 p_3\right) q(x)\right] T_{\mathrm{M}_0} = \int \mathrm{d}^3x\, q^{+}(x)[\alpha_3 p_3 + (m^2 + p_{\perp}^2)^{1/2}\beta] q(x) = T_{\mathrm{M}}^{+} H T_{\mathrm{M}}.$$

Thus, we get

$$S_{\mathrm{M}}^{+} H S_{\mathrm{M}} = \int \mathrm{d}^3x\, q^{+}(x)\left(\frac{\boldsymbol{\alpha}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}}(m^2 + p_{\perp}^2)^{1/2} + \alpha_3 p_3\right) q(x) \tag{13}$$

where

$$\begin{aligned} S_{\mathrm{M}} &= T_{\mathrm{M}} T_{\mathrm{M}_0}^{+} \\ &= \exp\left[\!\left[-\frac{\mathrm{i}}{2} \int \mathrm{d}^3x\, q^{+}(x)\left\{\frac{\boldsymbol{\gamma}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}}\left[\tan^{-1}\left(\frac{p_{\perp}}{m}\right) - \frac{\pi}{2}\right]\right\} q(x)\right]\!\right] \\ &= \exp\left[+\frac{\mathrm{i}}{2} \int \mathrm{d}^3x\, q^{+}(x)\left(\frac{\boldsymbol{\gamma}_{\perp} \cdot \boldsymbol{p}_{\perp}}{p_{\perp}} \tan^{-1}\frac{m}{p_{\perp}}\right) q(x)\right] \end{aligned} \tag{14}$$

which is the CT analogue of the Melosh transformation T_{M}. In passing, we remark that S_{M}, like T_{M}, is also a *good* operator in the sense that it does not contain p_3 in the exponent and is invariant only under the two-dimensional rotation group. However, since S_{M} retains the $\boldsymbol{\alpha}_{\perp}$ parts of the Hamiltonian, we may perhaps expect it to be of some use in a different context. For example, we may note that in the high-energy WK representation

$$\alpha_3=\sigma_3\times\sigma_k,\qquad \gamma_k=\sigma_2\times\sigma_k,\qquad \hat{\sigma}_k=I\times\sigma_k\qquad\text{and}\qquad \beta=\gamma_4=-\sigma_1\times I \tag{15}$$

where σ_k and I are the usual Pauli matrices, and $\hat{\sigma}_k$ are the Dirac spin matrices, the transformed quark Hamiltonian (13) breaks up into a direct sum of diagonal blocks and this is obviously desirable from a computational point of view.

4. The CT analogue of the Garrido–Pascual transformation

As a second application of the formula (14) we now obtain the CT analogue of the GP transformation (Garrido and Pascual 1959). In the theory (Bhabha 1945) of an elementary particle of arbitrary spin described by the relativistic wave equation

$$(\beta_\mu\partial_\mu+m)\psi=0, \tag{16}$$

where the β_μ are assumed to satisfy the commutation relation

$$[\beta_\lambda,[\beta_\mu,\beta_\nu]]=\delta_{\lambda\mu}\beta_\nu-\delta_{\lambda\nu}\beta_\mu, \tag{17}$$

Case (1954) has proposed the *appropriate* Hamiltonian to be

$$H=\alpha_k p_k+\beta m;\qquad \alpha_k=-\mathrm{i}[\beta_k,\beta_4]. \tag{18}$$

It was shown by Garrido and Pascual (1959) that this Hamiltonian may be diagonalised by the FW-type transformation

$$T_g=\exp\left(-\mathrm{i}g\frac{\beta_k p_k}{p}\tan^{-1}\frac{p}{m}\right);\qquad T_g^+HT_g=(p^2+m^2)^{1/2}\beta \tag{19}$$

where g is the spin of the particle. We may note that (16), (17), (18) and (19) reduce to the corresponding equations of the Dirac and Kemmer (1939) particles when we let $g=\frac{1}{2}$ and $g=1$ respectively. In exactly the same manner as for the Dirac case, we obtain the CT analogue S_g of T_g:

$$S_g=\exp\left(\mathrm{i}g\frac{\beta_k p_k}{p}\tan^{-1}\frac{m}{p}\right) \tag{20}$$

and S_g sends the Case Hamiltonian (18) into

$$S_g^+HS_g=\frac{\alpha_k p_k}{p}(m^2+p^2)^{1/2}. \tag{21}$$

In the Kemmer case of $g=1$, we get

$$S_1=1+\frac{im}{E}\frac{\beta_k p_k}{p}-\left(1-\frac{p}{E}\right)\left(\frac{\beta_k p_k}{p}\right)^2;\qquad E=(p^2+m^2)^{1/2}. \tag{22}$$

5. Simultaneous eigenstates of energy and helicity in the Dirac theory

Next, we wish to make some remarks on the use of T and S specifically in the case of the Dirac equation. The discussion of the free-particle FW transformation (3) generally found in the literature (Foldy 1962, Rose 1961, Bjorken and Drell 1964, Schweber 1964, Messiah 1962, Roman 1965, Eisele 1969, Davydov 1965, Umezawa 1956) seems to be confined to showing that in the Dirac Hamiltonian (1) the positive and negative energy solutions are separately described by two component equations if β is diagonal and to the circumstance that T enables one to give satisfactory definitions of mean operators like position, velocity and spin. The observation made about the CT transformation on the other hand is that it transforms away the β term in (1) leading to a representation in which the helicity eigenstates get decoupled.

We need, in what follows, the result that the helicity operator $\Sigma = \hat{\sigma}_k p_k/p$ commutes with both the transformations T and S. Although this is implied in the work of Jacob and Wick (1959), we give here a simple proof. We recall that the well known relation $[\Sigma, H] = 0$ is usually proved by showing that Σ commutes separately with β and $\alpha_k p_k$. Precisely because of this, Σ commutes also with their product $\beta\alpha_k p_k = \mathrm{i}\gamma_k p_k$ and hence with S as well as T, i.e.,

$$[\Sigma, T] = 0, \qquad [\Sigma, S] = 0. \tag{23}$$

This invariance of Σ under both the FW and CT transformations immediately gives a very simple procedure for the construction of the simultaneous eigenstates of energy and helicity of the free Dirac particle, both in the standard DP representation

$$\alpha_k = \sigma_1 \times \sigma_k; \qquad \gamma_k = \sigma_2 \times \sigma_k; \qquad \hat{\sigma}_k = I \times \sigma_k; \qquad \beta = \gamma_4 = \sigma_3 \times I$$

and in the high-energy WK representation (15).

We observe first of all that the helicity operator Σ is the same in both the representations, and helicity eigenstates in either are given by the columns of the matrix

$$V = \begin{pmatrix} \xi & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{pmatrix}$$

where

$$\xi = N\begin{pmatrix} p + p_3 \\ p_+ \end{pmatrix}; \qquad \eta = N\begin{pmatrix} -p_- \\ p + p_3 \end{pmatrix}; \qquad N = [2p(p + p_3)]^{-1/2}$$

and $p_\pm = p_1 \pm \mathrm{i}p_2$.

The Hamiltonian and the helicity are transformed by T and S repectively into

$$T^+HT = (p^2 + m^2)^{1/2}\beta; \qquad T^+\Sigma T = \Sigma$$

$$S^+HS = \frac{\alpha_k p_k}{p}(p^2 + m^2)^{1/2}; \qquad S^+\Sigma S = \Sigma$$

Since in the DP representation β is diagonal, and in the WK representation α_k are block-diagonal, the columns of V would be the simultaneous eigenstates of energy and helicity in both the transformed bases with the appropriate representations. Therefore these simultaneous eigenstates in the original bases are simply given by the columns of $U = TV$ in the DP representation and $U = SV$ in the WK representation.

We now show that the transformation S (or T) coupled with the rotation (Merzbacher 1970)

$$R = \exp\left(\frac{\mathrm{i}}{2} \frac{\hat{\sigma}_1 p_2 - \hat{\sigma}_2 p_1}{(p_1^2 + p_2^2)^{1/2}} \cos^{-1}\frac{p_3}{p}\right)$$

gives a method of constructing the simultaneous eigenstates of H and Σ in any *arbitrary* representation of the Dirac matrices. On observing that

$$S^+HS=(p^2+m^2)^{1/2}\frac{\alpha_k p_k}{p};\qquad S^+\Sigma S=\Sigma;\qquad R^+\frac{\alpha_k p_k}{p}R=\alpha_3;\qquad R^+\Sigma R=\hat{\sigma}_3$$

$$T^+HT=(p^2+m^2)^{1/2}\beta;\qquad T^+\Sigma T=\Sigma;\qquad R^+\beta R=\beta;\qquad R^+\Sigma R=\hat{\sigma}_3,$$

it is easy to see that if W is the matrix of the simultaneous eigenstates of $\hat{\sigma}_3$ and α_3 (or β), $V=RW$ is the matrix of simultaneous eigenstates of Σ and $\alpha_k p_k/p$ (or β). If we now transform by S (or T) we immediately get $U=SV=SRW$ (or $U=TV=TRW$) as the desired matrix of simultaneous eigenstates of Σ and H in any arbitrary representation, as W can almost always be obtained by inspection. For example, in the DP and WK representations W is simply the unit matrix.

Acknowledgments

We are grateful to one of the referees for his kind interpretation of the meaning of the separation of positive and negative helicity states which result from a CT transformation.

References

Bhabha H J 1945 *Rev. Mod. Phys.* **17** 200–16
Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill) pp 46–8
Bose S K, Gamba A and Sudarshan E C G 1959 *Phys. Rev.* **113** 1661–3
Case K M 1954 *Phys. Rev.* **95** 1323–8
Cini M and Touscheck B 1958 *Nuovo Cim.* **7** 422–3
Davydov A S 1965 *Quantum Mechanics* (London: Pergamon) pp 230–2
Eisele J A 1969 *Modern Quantum Mechanics with Applications to Elementary Particle Physics* (New York: Wiley-Interscience) pp 362–88
Foldy L L 1962 *Quantum Theory III* ed. D R Bates (New York: Academic) pp 20–3
Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29–36
Garrido L M and Pascual P 1959 *Nuovo Cim.* **12** 181–90
Jacob M and Wick G C 1959 *Ann. Phys., NY* **7** 404–28
Kemmer N 1939 *Proc. R. Soc.* A **173** 91–116
Krajcik R A and Nieto M M 1976 *Phys. Rev.* **D 14** 418–36
Melosh H J 1973 *PhD Thesis* California Institute of Technology
Merzbacher E 1970 *Quantum Mechanics* (New York: Wiley) pp 581–3
Messiah A 1962 *Quantum Mechanics* vol. 2 (Amsterdam: North-Holland) pp 941–4
Roman P 1965 *Advanced Quantum Theory* (Reading, Mass.: Addison-Wesley) pp 133–6
Rose M E 1961 *Relativistic Electron Theory* (New York: Wiley) pp 87–92
Schweber S S 1964 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row) pp 91–5
Umezawa H 1956 *Quantum Field Theory* (Amsterdam: North-Holland) pp 45–7
Weyers J 1975 *Constituent Quarks and Current Quarks in Particle Interactions at very High Energies* eds D Speiser, F Halzen and J Weyers (New York: Plenum) part B, pp 43–95